GREEN FUNCTIONS FOR CLASSICAL EUCLIDEAN MAXWELL THEORY

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Summary. - Recent work on the quantization of Maxwell theory has used a non-covariant class of gauge-averaging functionals which include explicitly the effects of the extrinsic-curvature tensor of the boundary, or covariant gauges which, unlike the Lorentz case, are invariant under conformal rescalings of the background four-metric. This paper studies in detail the admissibility of such gauges at the classical level. It is proved that Euclidean Green functions of a second- or fourth-order operator exist which ensure the fulfillment of such gauges at the classical level, i.e. on a portion of flat Euclidean four-space bounded by three-dimensional surfaces. The admissibility of the axial and Coulomb gauges is also proved.

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1. - Introduction.

The recent progress on Euclidean quantum gravity on manifolds with boundary [1] has led to a detailed investigation of mixed boundary conditions for gauge fields and gravitation. These have been applied to the one-loop semiclassical evaluation of quantum amplitudes which are motivated by the analysis of the wave function of the universe. In particular, we are here interested in the quantization programme for Euclidean Maxwell theory. If one uses Faddeev-Popov formalism, the above amplitudes involve Gaussian averages over gauge functionals $\Phi(A)$ which ensure that both the operator on perturbations of A_b (A_b being the electromagnetic potential) and the ghost operator admit well-defined Feynman Green functions for the given boundary conditions. As is well known, the gauge-averaging term $\frac{1}{2\alpha}[\Phi(A)]^2$ (α being a dimensionless parameter) modifies the second-order operator on A_b that one would obtain from the Maxwell Lagrangian $\frac{1}{4}F_{ab}F^{ab}$. For example, the Lorentz choice $\Phi_L(A) = \nabla^b A_b$ leads to the following second-order operator on A_b perturbations (the background value of A_b is set to zero):

$$Q^{bc} = -g^{bc}\Box + R^{bc} + \left(1 - \frac{1}{\alpha}\right)\nabla^b\nabla^c \ . \tag{1.1}$$

With a standard notation, g is the background four-metric, ∇^b denotes covariant differentiation with respect to the Levi-Civita connection of the background, $\Box$ is defined as $g^{ab}\nabla_a\nabla_b$ ($-\Box$ is the Laplacian having positive spectrum on compact manifolds), and R^{bc} is the Ricci tensor of the background. The Feynman choice for α: $\alpha = 1$, reduces Q^{bc} to the standard Hodge-de Rham operator on one-forms.

In the underlying *classical* theory, one starts from the purely Maxwell Lagrangian (but bearing in mind that the metric is positive-definite, unlike the Lorentzian case), which is invariant under (infinitesimal) gauge transformations of the kind

$$^{f}A_b = A_b + \nabla_b f \ , \tag{1.2}$$

with the function f being freely specifiable. However, when a gauge condition is imposed:

$$\Phi(A) = 0 \ , \tag{1.3}$$

this leads to non-trivial changes in the original scheme. They are as follows.

(i) The preservation of eq. (1.3), viewed as a constraint, has the effect of turning the original first-class constraints into the second-class [2].

(ii) The function f has to obey a differential equation, instead of being freely specifiable.

For example, suppose one starts from a potential A_b which does not obey the Lorentz gauge:

$$\nabla^b A_b \neq 0 \; . \tag{1.4}$$

Nevertheless, it is possible to ensure that the gauge-transformed potential ${}^f A_b$ does actually obey the Lorentz gauge $\nabla^b({}^f A_b) = 0$, provided that f satisfies the second-order equation

$$\Box f = -\nabla^b A_b \; . \tag{1.5}$$

Denoting by $\mathcal{G}$ the inverse of the $\Box$ operator, i.e. its Green's function, the solution of eq. (1.5) is formally expressed as

$$f = -\mathcal{G} \; \nabla^b A_b \; . \tag{1.6}$$

For this scheme to hold one has to prove that, with the given boundary conditions, which are mixed for gauge fields [1], the inverse of the $\Box$ operator actually exists and can be constructed explicitly.

For this purpose, sect. **2** constructs the Green's function for the non-covariant gauges proposed in refs.[3–5], and sect. **3** performs a similar analysis for the conformally invariant gauge of Eastwood and Singer [6]. Sections **4** and **5** deal with the axial and Coulomb gauges, respectively. Concluding remarks are presented in sect. **6**.

2. - Green functions with non-covariant gauges.

This section provides the classical counterpart of the analysis in refs.[3–5]. Hence we consider a portion of flat Euclidean four-space bounded by the three-surfaces Σ_1 and Σ_2, say. The region in between Σ_1 and Σ_2 is foliated by three-dimensional hypersurfaces with

extrinsic-curvature tensor K. Denoting by β a dimensionless parameter, the gauges we impose read

$$\nabla^c A_c - \beta A_0 \mathrm{Tr} K = 0 \ . \tag{2.1}$$

These gauges reduce to the Lorentz choice if β is set to zero, and their consideration is suggested by the need to characterize the most general class of gauge conditions for Euclidean Maxwell theory in the presence of boundaries. In this respect, they are still a particular case of a more general family of gauges, as shown in ref.[5]. In the quantum theory, they have the advantage of being the only relativistic gauges for which the one-loop semiclassical theory can be explicitly evaluated [5], despite the non-covariant nature resulting from the βA_0 TrK term.

In the classical theory, the first problem is to make sure that equation (2.1) can be actually satisfied. This problem was not even addressed in [3–5], and is the object of our first investigation. Following the example given in the introduction, let us assume that the original potential A_b (a connection one-form in geometric language) does not satisfy (2.1). After the gauge transformation (1.2), the potential ${}^f A_b$ satisfies (2.1) if and only if f obeys the following equation:

$$\left(\Box - \beta \mathrm{Tr} K \frac{\partial}{\partial \tau} \right) f = - \Big(\nabla^c A_c - \beta A_0 \mathrm{Tr} K \Big) \ . \tag{2.2}$$

In eq. (2.2) we denote by τ a radial coordinate, and hence we are assuming that concentric three-sphere boundaries are studied, as in [4,5]. In the system of local coordinates appropriate for the case when flat Euclidean four-space is bounded by concentric three-spheres, eq. (2.2) takes the form

$$\left[\frac{\partial^2}{\partial \tau^2} + \frac{3(1-\beta)}{\tau} \frac{\partial}{\partial \tau} + \frac{1}{\tau^2} {}^{(3)}\nabla_i {}^{(3)}\nabla^i \right] f = - \Big(\nabla^c A_c - \beta A_0 \mathrm{Tr} K \Big) \ , \tag{2.3}$$

where ${}^{(3)}\nabla_i$ denotes three-dimensional covariant differentiation tangentially with respect to the Levi-Civita connection of the boundary. Our first problem is now to find the Green's

function $\widetilde{\mathcal{G}}$ of the operator in square brackets in eq. (2.3), so that the function f can be expressed as

$$f = -\widetilde{\mathcal{G}}\Big(\nabla^c A_c - \beta A_0 \mathrm{Tr} K\Big) . \tag{2.4}$$

The general solution of eq. (2.3) is given by the general solution f_0 of the homogeneous equation plus a particular solution of the full equation. The solution f_0 can always be found, since the homogeneous equation is a second-order differential equation with a regular singular point at $\tau = 0$ (indeed, in our two-boundary problem, τ lies in the closed interval $[a, b]$, with $a > 0$). It is convenient to study a mode-by-mode form of eq. (2.3), i.e. the infinite number of second-order equations resulting from the expansion of f on a family of concentric three-spheres:

$$f(x, \tau) = \sum_{n=1}^{\infty} f_n(\tau) Q^{(n)}(x) , \tag{2.5}$$

where $Q^{(n)}(x)$ are the scalar harmonics on S^3, in the local coordinates x. This leads to the equations

$$\left[\frac{d^2}{d\tau^2} + \frac{3(1-\beta)}{\tau}\frac{d}{d\tau} - \frac{(n^2-1)}{\tau^2}\right] f_n = -\Big(\nabla^c A_c - \beta A_0 \mathrm{Tr} K\Big)_n , \tag{2.6}$$

for all integer values of $n \geq 1$. The problem of inverting a second-order operator and finding f is therefore reduced to solving for f_n, for all $n \geq 1$. For this purpose, it is appropriate to take an integral transform of both sides of eq. (2.6), and then anti-transform to obtain f_n, for all $n \geq 1$. The integral transform should turn the left-hand side of eq. (2.6) into the product of a polynomial with the transform of $f_n(\tau)$. To overcome the technical difficulties resulting from the negative powers of τ in the operator, we define a new variable

$$w \equiv \log(\tau) , \tag{2.7}$$

so that our problem becomes the one of finding the integral transform of

$$\left[\frac{d^2}{dw^2} + (2-3\beta)\frac{d}{dw} - (n^2-1)\right] f_n(w)$$
$$= -\mathrm{e}^{2w}\Big(\nabla^c A_c - \beta A_0 \mathrm{Tr} K\Big)_n . \tag{2.8}$$

If we extend the definition of $f_n(w)$, requiring that it should vanish for $w < \log(a)$ and $w > \log(b)$, we can define its Fourier transform

$$\mathcal{F}(f_n(w)) \equiv \widetilde{f}_n(p) \equiv \int_{-\infty}^{\infty} f_n(w) \mathrm{e}^{-ipw} dw \ . \tag{2.9}$$

Denoting by Ω_n the right-hand side of eq. (2.8), one thus finds (hereafter $s \equiv ip$)

$$f_n(w) = \int_{-\infty}^{\infty} \frac{\mathcal{F}(\Omega_n)\mathrm{e}^{sw}}{[s^2 + (2-3\beta)s - (n^2-1)]} ds \ , \tag{2.10}$$

up to a multiplicative constant which is unessential for our purposes. Of course, the contour in eq. (2.10) has been rotated to integrate over s. In eq. (2.10), the integrand has poles corresponding to the zeros of the equation

$$s^2 + (2-3\beta)s - (n^2-1) = 0 \ . \tag{2.11}$$

This equation has two real solutions (corresponding to purely imaginary values of p) given by

$$s = \frac{(3\beta-2) \pm \sqrt{(3\beta-2)^2 + 4(n^2-1)}}{2} \ . \tag{2.12}$$

This means that the integral formula for $f_n(w)$ should be regarded as a contour integration, where the complex contour goes around the real roots of eq. (2.11). The various possible prescriptions are the counterpart of the Green functions of quantum field theory, i.e. retarded, advanced, Feynman, Wightman, Hadamard etc. [7]. Note, however, that our analysis is one-dimensional and entirely classical. The solution of eq. (2.3) reads therefore

$$f(x,\tau) = \sum_{i=1}^{2} \sum_{n=1}^{\infty} a_n^{(i)} \ \tau^{\rho_n^{(i)}} Q^{(n)}(x) + \sum_{n=1}^{\infty} f_n(w(\tau)) Q^{(n)}(x) \ , \tag{2.13}$$

where $a_n^{(i)}$ are constant coefficients and $\rho_n^{(i)}$ denotes, for $i = 1, 2$, the roots (2.12).

Interestingly, the value $\beta = \frac{2}{3}$, which was found to play an important role in the one-loop semiclassical analysis of ref.[5], emerges naturally in our case as the particular value of β for which the denominator has real roots $\pm\sqrt{n^2-1}$. When $n = 1$, the integrand in eq.

(2.10) has a double pole at $s = 0$. This corresponds to the lack of a regular decoupled mode for the normal component of the electromagnetic potential in the one-boundary problem, in the quantum theory [5].

3. - Green's function in the Eastwood-Singer gauge.

The gauge conditions studied in the classical theory of the electromagnetic field are not, in general, invariant under conformal rescalings of the metric. For example, not even the Lorentz gauge is preserved under the conformal rescaling $\widetilde{g}_{ab} = \Omega^2 \ g_{ab}$ [6]. This is not entirely desirable, since we know that the equations of vacuum Maxwell theory are conformally invariant. However, as shown in ref.[6], an operator on one-forms can be defined in such a way that the resulting gauge condition is conformally invariant. This reads

$$\nabla_b \left[\left(\nabla^b \nabla^c - 2R^{bc} + \frac{2}{3} R g^{bc} \right) A_c \right] = 0 \ , \tag{3.1}$$

where R is the trace of the Ricci tensor. Suppose now that the original potential A_c does not satisfy eq. (3.1). The gauge-transformed potential obtained according to (1.2) will instead satisfy eq. (3.1) provided that f obeys the fourth-order equation

$$\begin{aligned} &\left[\Box^2 + \nabla_b \left(-2R^{bc} + \frac{2}{3} R g^{bc} \right) \nabla_c \right] f \\ &= -\nabla_b \left[\left(\nabla^b \nabla^c - 2R^{bc} + \frac{2}{3} R g^{bc} \right) A_c \right] \ , \end{aligned} \tag{3.2}$$

where $\Box^2$ denotes the $\Box$ operator composed with itself, i.e. $g^{ab} g^{cd} \nabla_a \nabla_b \nabla_c \nabla_d$. Our problem is now to find the Green's function of the fourth-order operator on the left-hand side of eq. (3.2). Since our paper is restricted to the analysis of flat Euclidean backgrounds bounded by concentric three-spheres, we need to invert the operator [8]

$$\begin{aligned} \Box^2 \equiv & \frac{\partial^4}{\partial \tau^4} + \frac{6}{\tau} \frac{\partial^3}{\partial \tau^3} + \frac{3}{\tau^2} \frac{\partial^2}{\partial \tau^2} - \frac{3}{\tau^3} \frac{\partial}{\partial \tau} \\ & + \frac{2}{\tau^2} \left(\frac{\partial^2}{\partial \tau^2} + \frac{1}{\tau} \frac{\partial}{\partial \tau} \right) {}^{(3)}\nabla_i {}^{(3)}\nabla^i + \frac{1}{\tau^4} \left({}^{(3)}\nabla_i {}^{(3)}\nabla^i \right)^2 \ , \end{aligned} \tag{3.3}$$

which ensures the fulfillment of the gauge condition $\Box \nabla^c A_c = 0$. As in sect. **2**, we begin by taking the Fourier transform of eq. (3.2) when the curvature of the background vanishes, and a mode-by-mode analysis is performed. We therefore consider the fourth-order operator [8]

$$\Box_n^2 \equiv \frac{d^4}{d\tau^4} + \frac{6}{\tau}\frac{d^3}{d\tau^3} - \frac{(2n^2-5)}{\tau^2}\frac{d^2}{d\tau^2} - \frac{(2n^2+1)}{\tau^3}\frac{d}{d\tau} + \frac{(n^2-1)^2}{\tau^4} . \tag{3.4}$$

The introduction of the variable (2.7) leads to a remarkable cancellation of some derivatives. Hence one finds the equation

$$\begin{aligned} &\left[\frac{d^4}{dw^4} - 2(n^2+1)\frac{d^2}{dw^2} + (n^2-1)^2\right] f_n(w) \\ &= -e^{4w}\left(\Box \nabla^c A_c\right)_n . \end{aligned} \tag{3.5}$$

We now take the Fourier transform of eq. (3.5), and then anti-transform setting $s \equiv ip$. This leads to the contour formula (cf. eq. (2.10))

$$f_n(w) = \int_{-\infty}^{\infty} \frac{\mathcal{F}(\widetilde{\Omega}_n)e^{sw}}{[s^4 - 2(n^2+1)s^2 + (n^2-1)^2]} ds , \tag{3.6}$$

where $\widetilde{\Omega}_n$ is the right-hand side of eq. (3.5). The structure of the poles of the integrand differs substantially from the one found in sect. **2**. The poles are now the roots of the fourth-order algebraic equation

$$s^4 - 2(n^2+1)s^2 + (n^2-1)^2 = 0 , \tag{3.7}$$

which has the four real roots $\pm(n \pm 1)$ [8].

In our flat background with boundary, the general solution of eq. (3.2) is then given by

$$f(x,\tau) = u(x,\tau) + \sum_{n=1}^{\infty} f_n(w(\tau)) Q^{(n)}(x) , \tag{3.8}$$

where $u(x,\tau)$ solves the homogeneous equation, and can be written as

$$u(x,\tau) = \sum_{i=1}^{4} \sum_{n=1}^{\infty} b_n^{(i)} \; \tau^{\sigma_n^{(i)}} \; Q^{(n)}(x) \; , \tag{3.9}$$

where $\sigma_n^{(i)}$ denotes, for $i = 1, 2, 3, 4$, a solution of eq. (3.7), and $b_n^{(i)}$ are constant coefficients.

4. - Axial gauge.

Covariant gauges are not the only possible choice for the quantization of Maxwell theory. By contrast, non-covariant gauges play an important role as well, and a comprehensive review may be found in ref.[9]. In our paper we focus on the axial and Coulomb gauges at the classical level. Let n^a be the normal to the hypersurface Σ belonging to the family of hypersurfaces foliating the portion of flat Euclidean four-space bounded by Σ_1 and Σ_2. If A_b does not obey the axial gauge:

$$n^b A_b \neq 0 \; , \tag{4.1}$$

one can nevertheless perform the transformation (1.2) and then look for a function f such that

$$n^b (A_b + \nabla_b f) = 0 \; . \tag{4.2}$$

We can safely assume that n^a takes the form $(1, 0, 0, 0)$, while Σ_1 and Σ_2 consist (once again) of concentric three-spheres. In such a case, eq. (4.2) reduces to

$$\frac{\partial f}{\partial \tau} = -A_0(x,\tau) \; , \tag{4.3}$$

which can be solved for f in the form

$$f(x,\tau) = f(x,a) - \int_a^{\tau} A_0(x,y) dy \; , \tag{4.4}$$

for all $\tau \in [a, b]$, where a and b are the radii of the three-spheres Σ_1 and Σ_2, respectively.

5. - Coulomb gauge.

The Coulomb gauge is a non-covariant gauge which involves the three-dimensional divergence of the three-dimensional vector potential [1]. By this we mean that tangential covariant derivatives of A_i are taken with respect to the Levi-Civita connection of the induced metric on the boundary. These three-dimensional covariant derivatives are denoted by ${}^{(3)}\nabla_i$ in our paper, and more often by $|_i$ in the literature on general relativity. If A_i does not satisfy the Coulomb gauge, we consider the tangential components of the gauge transformation (1.2), and we require that

$$ {}^{(3)}\nabla^i \Big(A_i + {}^{(3)}\nabla_i f \Big) = 0 \ . \tag{5.1} $$

The tangential components A_i admit a standard expansion on a family of concentric three-spheres according to [1]

$$ A_i(x,\tau) = \sum_{n=2}^{\infty} \Big[\omega_n(\tau) S_k^{(n)}(x) + g_n(\tau) P_k^{(n)}(x) \Big] \ , \tag{5.2} $$

where $S_k^{(n)}(x)$ and $P_k^{(n)}(x)$ are transverse and longitudinal harmonics on S^3, respectively [1]. Bearing in mind that [1]

$$ S_k^{(n) \ |k}(x) = 0 \ , \tag{5.3} $$

$$ P_k^{(n) \ |k}(x) = -Q^{(n)}(x) \ , \tag{5.4} $$

eq. (5.1) takes the form

$$ \sum_{n=2}^{\infty} \Big[(n^2-1) f_n(\tau) + g_n(\tau) \Big] Q^{(n)}(x) = 0 \ , \tag{5.5} $$

which implies that

$$ f_n(\tau) = -\frac{g_n(\tau)}{(n^2-1)} \ \text{ for all } n \geq 2 \ , \tag{5.6} $$

while $f_1(\tau)$ remains freely specifiable. The classical *gauge* mode f_1 is not fixed by the condition (5.1), since its coefficient vanishes (it equals n^2-1 evaluated at $n=1$). The

result (5.6) can be now inserted into the expansion (2.5) to express the gauge function as an infinite sum of modes proportional to longitudinal modes. Hence the f_n modes obey the same boundary conditions imposed on longitudinal modes of A_i. The residual gauge freedom encoded by $f_1(\tau)$, however, deserves further investigation (cf. the quantum analysis in section 7.8 of ref.[1]).

6. - Concluding remarks.

Our paper has studied the admissibility of non-covariant or conformally invariant gauge conditions for classical Maxwell theory in the presence of boundaries. Since the background is flat Euclidean (instead of flat Minkowskian), our analysis provides the classical counterpart of the Euclidean version of the quantum theory. We might have addressed the simpler problem of imposing that both the original potential and the gauge-transformed one (see (1.2)) satisfy the gauge condition. We have instead studied the more difficult case, where the gauge condition is eventually fulfilled after performing the transformation (1.2). The function f is no longer freely specifiable, but obeys second- or fourth-order equations, whose Green's functions have been explicitly evaluated in sects. **2** and **3**. Moreover, sects. **4** and **5** show that, when the axial or Coulomb gauge are imposed, it is simpler to solve for the gauge function f. Equation (4.4) provides an integral representation, while (2.5) and (5.6) yield an infinite sum for f, with modes proportional to the longitudinal modes for A_i.

Our analysis puts on solid ground the consideration of the gauge conditions (2.1) and (3.1) for classical Euclidean Maxwell theory on manifolds with boundary. The consideration of boundary effects is of crucial importance for a correct formulation of the classical and quantum theories: potential theory, Casimir effect and one-loop quantum cosmology are some of the many examples one may provide [1]. It now remains to be seen how to include the effects of curvature (of the background) in the evaluation of our Green's functions. Moreover, a deep problems consists in studying the possible occurrence of Gribov phenomena if eqs. (2.1) and (3.1) are applied to the quantization of non-Abelian gauge theories [10].

On the quantum side, further developments are likely to occur in the analysis of the one-loop effective action in non-covariant [11] or covariant [12] gauges, and in the application of such techniques to the evaluation of Casimir energies [13]. If such a programme could be accomplished, it would provide further evidence in favour of quantum cosmology having a deep influence on current developments in the theory of quantized gauge fields [1].

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